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Hermite–Fejér Interpolation with Boundary Conditions for ρ-Normal Sets

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The $H_{npq}f$ polynomials are extensions of the generalized Hermite–Fejér interpolating polynomials, H_nf, in that they incorporate boundary conditions. For such polynomials, one can define (p, q)–ρ-normal sets which correspond to the ρ-normal sets for H_nf. It is shown that the sequence $\{H_{npq}f\}$ based on a (p, q)–ρ-normal set converges uniformly to f for all continuous f.

1. INTRODUCTION

The Hermite–Fejér (HF) polynomials H_nf were introduced by Fejér in 1916 [1] as a means to prove the Weirstrass approximation theorem constructively using interpolating polynomials rather than the approximating Bernstein polynomials. They are defined in terms of a triangular set of points

$$T := \{x_{kn} : k = 1, ..., n; n = 1, 2, ...; x_{jn} < x_{kn} \text{ if } j < k\}$$

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contained in $I := [-1, 1]$ as follows

$$H_n f(x) := \sum_{k=1}^{n} f_{kn} h_{kn}(x) \qquad (f_{kn} := f(x_{kn})),$$

where

$$h_{kn}(x) := v_{kn}(x)\, l_{kn}^2(x)$$
$$l_{kn}(x) := \omega_n(x)/((x - x_{kn})\, \omega_n'(x_{kn}))$$
$$\omega_n(x) := \prod_{j=1}^{n} (x - x_{jn})$$

and

$$v_{kn}(x) := 1 - \omega_n''(x_{kn})(x - x_{kn})/\omega_n'(x_{kn})$$

so that

$$h_{kn}(x_{jn}) = \delta_{jk},\ h_{kn}'(x_{jn}) = 0, \qquad j, k = 1, ..., n.$$

It follows that the HF polynomials $H_n f$ satisfy the conditions

$$H_n f(x_{kn}) = f_{kn}, \qquad H_n f'(x_{kn}) = 0, \quad k = 1, ..., n.$$

Fejér [1] showed that if

$$x_{n-k+1,n} = \cos\frac{2k-1}{2n}\pi, \qquad k = 1, ..., n;\ n = 1, 2, ...$$

then, for all $f \in C(I)$,

$$\|f - H_n f\|_I = o(1) \qquad \text{as} \quad n \to \infty,$$

where, for any interval J,

$$\|g\|_J := \max_{x \in J} |g(x)|.$$

Since Fejér's original paper, the HF and related polynomials have been studied quite extensively as is shown by a recent bibliography [4] with over 350 titles. The generalizations and extensions of $H_n f$ have been in various directions. In one direction they have culminated in the H_{npq} polynomials which are based on a triangular set of points $T \subset (-1, 1)$ and include boundary conditions at the endpoints of I. They are defined for nonnegative integers p and q in terms of the vectors $\mathbf{d}_n$, $\mathbf{e}_n$, and $\mathbf{m}_n$ of lengths n, p, and q respectively, as

$$\begin{aligned} H_{npq}(f, \mathbf{d}_n, \mathbf{e}_n, \mathbf{m}_n; x) &:= \sum_{k=1}^{n} f_{kn} h_{knpq}(x) + \sum_{k=1}^{n} d_{kn} \hat{h}_{knpq}(x) \\ &\quad + \sum_{s=0}^{p-1} e_{s+1,n} \chi_{snpq}(x) + \sum_{t=0}^{q-1} m_{t+1,n} \bar{\chi}_{tnpq}(x), \end{aligned} \tag{1}$$

where

$$h_{knpq}(x) := v_{knpq}(x)\, A_{knpq}(x) \tag{2}$$

$$A_{knpq}(x) := \left(\frac{1-x}{1-x_{kn}}\right)^p \left(\frac{1+x}{1+x_{kn}}\right)^q l_{kn}^2(x) \tag{3}$$

$$v_{knpq}(x) := 1 + \left[\frac{p}{1-x_{kn}} - \frac{q}{1+x_{kn}} - \frac{\omega_n''(x_{kn})}{\omega_n'(x_n)}\right](x - x_{kn}) \tag{4}$$

$$\hat{h}_{knpq}(x) := (x - x_{kn})\, A_{knpq}(x) \tag{5}$$

$$\chi_{snpq}(x) := V_{snpq}(x) \frac{(1-x)^s}{s!} \left(\frac{\omega_n(x)}{\omega_n(1)}\right)^2 \left(\frac{1+x}{2}\right)^q \tag{6}$$

$$\bar{\chi}_{tnpq}(x) := \bar{V}_{tnpq}(x) \frac{(1+x)^t}{t!} \left(\frac{\omega_n(x)}{\omega_n(-1)}\right)^2 \left(\frac{1-x}{2}\right)^p \tag{7}$$

$$(-1)^s V_{snpq}(x) := \sum_{\sigma=0}^{p-1-s} a_{\sigma nq}(1-x)^\sigma \tag{8}$$

$$(-1)^t \bar{V}_{tnpq}(x) := \sum_{\tau=0}^{q-1-t} \bar{a}_{\tau np}(1+x)^\tau. \tag{9}$$

It has been shown by Knoop [6] that the coefficients $a_{\sigma nq}$ and $\bar{a}_{\tau np}$ in (8) and (9), which are chosen so that

$$\chi_{snpq}^{(i)}(1) = \delta_{is}, \qquad 0 \leqslant i, s \leqslant p-1$$
$$\bar{\chi}_{tnpq}^{(j)}(-1) = \delta_{jt}, \qquad 0 \leqslant j, t \leqslant q-1,$$

are independent of both p and s or q and t, respectively. When $p > 0$, $e_{1n} = f(1)$ and when $q > 0$, $m_{1n} = f(-1)$. Otherwise, the components of $\mathbf{d}_n$, $\mathbf{e}_n$, and $\mathbf{m}_n$ are arbitrary and may vary with n.

For certain special cases of $\mathbf{d}_n$, $\mathbf{e}_n$, and $\mathbf{m}_n$, we shall use the following notations: If $d_{kn} = f'(x_{kn})$, we shall write f' instead of $\mathbf{d}_n$. If $e_{sn} = f^{(s-1)}(1)$,

$s=1, \ldots, p$, we shall write $f^{(s)}$ instead of $\mathbf{e}_n$ while if e_{sn} does not depend on n, we shall write $\mathbf{e}$ instead of $\mathbf{e}_n$, and similarly for $\mathbf{m}_n$.

The H_{npq} polynomials have the interpolating properties

$$H_{npq}(x_{kn}) = f_{kn}, \qquad H'_{npq}(x_{kn}) = d_{kn},\ k=1, \ldots, n$$
$$H^{(s)}_{npq}(1) = e_{s+1,n},\ s=0, \ldots, p-1, \qquad H^{(t)}_{npq}(-1) = m_{t+1,n},\ t=0, \ldots, q-1$$

so that if $f \in P_{2n-1+p+q}$, where $\mathscr{P}_m$ is the set of all polynomials of degree $\leqslant m$, then $H_{npq}(f, f', f^{(s)}, f^{(t)}) = f$. In particular, if we set $f \equiv 1$, we get the important identity

$$\sum_{k=1}^{n} h_{knpq}(x) + \chi_{0npq}(x) + \bar{\chi}_{0npq}(x) = 1. \tag{10}$$

Special cases of the H_{npq} polynomials have been studied by many authors. Thus, the cases $p, q \in \{0, 2\}$ are the generalized HF polynomials while the case $p=q=1$ leads to the quasi-HF polynomial [8]. The general case was studied by Knoop [6] and Vértesi [11] when the points x_{kn} are the zeros of the Jacobi polynomial $P_n^{(\alpha, \beta)}$. They showed that if $\alpha \in [p-1, p)$ and $\beta \in [q-1, q)$, then

$$\|H_{npq}(f, \mathbf{d}_n, \mathbf{e}_n, \mathbf{m}_n) - f\|_I = o(1) \qquad \text{as} \quad n \to \infty \text{ for } f \in C \tag{11}$$

whenever

$$|d_{kn}| = \begin{cases} o(n/\log n) & \text{if } p-1 \leqslant \alpha \leqslant p-1/2,\ q-1 \leqslant \beta \leqslant q-1/2, \\ o(\min(n^{-2\alpha+2p}, n^{-2\beta+2q})) & \text{otherwise} \end{cases} \quad k=1, \ldots, n$$
$$|e_{sn}| = o(n^{2s-2}), \qquad s=2, \ldots, p$$
$$|m_{tn}| = o(n^{2t-2}), \qquad t=2, \ldots, q$$

and that (11) holds if $p-1.5 \leqslant \alpha < p$, $q-1.5 \leqslant \beta < q$, and $|\alpha - p - \beta + q| \leqslant 1$ provided that $d_{kn}=0$, $k=1, \ldots, n$, $\mathbf{e}_n = \mathbf{e}$, and $\mathbf{m}_n = \mathbf{m}$. It was also shown in [6] that for any $T \subset (-1, 1)$, $\chi_{0npq} \geqslant 0$, $\bar{\chi}_{0npq} \geqslant 0$ in I and that the coefficients $a_{\sigma nq}$ and $\bar{a}_{\tau np}$ in (8) and (9) depend only on the points x_{kn} and q or p, respectively, but not on p and s or q and t. We shall use the ideas in the proof of these facts in [6] to prove the following lemma:

LEMMA 1. *The coefficients $a_{\sigma nq}$ and $\bar{a}_{\tau np}$ in* (8) *and* (9) *are all non-negative.*

Proof. As in [6], we have that if $q=0$, χ'_{0npq} has $2n-1$ zeros in $[x_{1n}, x_{nn}]$ and $p-1$ zeros at $x=1$ and no others while if $q>0$, χ'_{0npq} has additional $q-1$ zeros at $x=-1$, one zero in $(-1, x_{1n})$, and no others.

Hence, if q is even, $\chi_{0npq} \to \infty$ as $x \to -\infty$ and if q is odd, $\chi_{0npq} \to -\infty$ as $x \to -\infty$. In either case, $V_{0npq} \to \infty$ as $x \to -\infty$ which implies that $a_{p-1,nq} \geqslant 0$. Since $a_{\sigma nq}$ is independent of p and s, we conclude that $a_{\sigma nq} \geqslant 0$ for all σ. Similarly $\bar{a}_{\sigma np} \geqslant 0$ for all τ.

COROLLARY 1. *For all $x \in I$*

$$|\chi_{snpq}(x)| \leqslant \frac{(1-x)^s}{s!} \chi_{0npq}(x), \qquad s = 1, ..., p-1 \tag{12}$$

$$|\bar{\chi}_{tnpq}(x)| \leqslant \frac{(1+x)^t}{t!} \bar{\chi}_{0npq}(x), \qquad t = 1, ..., q-1. \tag{13}$$

Proof. By Lemma 1, for all $x \in I$

$$|V_{snpq}(x)| \leqslant V_{0npq}(x), \qquad |\bar{V}_{tnpq}(x)| \leqslant \bar{V}_{0npq}(x).$$

Hence (12) and (13) follow from (6) and (7), respectively.

2. ρ-NORMAL SETS

In conjunction with his investigation of the convergence of HF interpolating polynomials, Fejér introduced the notions of normality and ρ-normality for triangular sets T.[1] The set T is said to be normal if $v_{kn} \geqslant 0$ in I and ρ-normal for some $\rho > 0$ if $v_{kn} \geqslant \rho$ in I for all k and n. Since $v_{kn}(x_{kn}) = 1$, it follows that $\rho \leqslant 1$. The importance of ρ-normality is that it ensures that the $\{H_n f\}$ is a sequence of positive operators in I. Fejér and Grünwald derived various properties of normal and ρ-normal sets which can be used to prove convergence results for HF interpolation. The outstanding example of ρ-normal sets is that given when the points x_{kn} are the zeros of $P_n^{(\alpha,\beta)}$ with $-1 < \alpha, \beta < 0$ in which case, $\rho = \min(-\alpha, -\beta)$. If $\alpha = 0$ or $\beta = 0$, the set is is only normal. Other ρ-normal sets are given in [10].

For the H_{npq} process, the appropriate generalizations of normality and ρ-normality will be called (p, q)-normality and (p, q)–ρ-normality and will be defined by the conditions that $v_{knpq}(x) \geqslant 0$ or $v_{knpq}(x) \geqslant \rho > 0$ as the case may be for all k and n and all $x \in I$. The particular case $(1, 1)$–ρ-normality was called quasi-ρ-normality and studied by Szász [8] and Sánta [7]. As in the ρ-normal case, the outstanding examples of (p, q)–ρ-normal sets are given by the zeros of $P_n^{(\alpha,\beta)}$ with $p-1 \leqslant \alpha < p$, $q-1 \leqslant \beta < q$. And as in the

[1] In [2], Fejér introduced properties A and B which are equivalent to normality and ρ-normality, respectively. In [3] he called point sets with property A normal and those with property B, normal in the strong sense. The term ρ-normality was coined by Grünwald [5].

normal case, we have that if a triangular set T is (p, q)-normal in I, then it is (p, q)–ρ-normal in $I_\varepsilon := [-1+\varepsilon, 1-\varepsilon]$, $0<\varepsilon<1$, with $\rho \geqslant \varepsilon/2$.

Clearly, in the (p, q)-normal case, we have from (10) that

$$\sum_{k=1}^{n} h_{knpq}(x) = \sum_{k=1}^{n} |h_{knpq}(x)| \leqslant 1 \qquad \text{in} \quad I \tag{14}$$

and that

$$\chi_{0npq}(x) \leqslant 1, \qquad \bar{\chi}_{0npq}(x) \leqslant 1 \text{ in } I. \tag{15}$$

These imply that for (p, q)–ρ-normal sets,

$$\sum_{k=1}^{n} A_{knpq}(x) \leqslant 1/\rho \qquad \text{in} \quad I \tag{16}$$

and using (12) and (13), that

$$|\chi_{snpq}(x)| \leqslant (1-x)^s/s! \qquad \text{in } I,\ s=0, ..., p-1 \tag{17}$$

$$|\bar{\chi}_{tnpq}(x)| \leqslant (1+x)^t/t! \qquad \text{in } I,\ t=0, ..., q-1. \tag{18}$$

Since we have convergence of the H_{npq} process for the (p, q)–ρ-normal set given by the zeros of $P_n^{(\alpha, \beta)}$ with $p-1 \leqslant \alpha < p$, $q-1 \leqslant \beta < q$, we may hope to have convergence for any (p, q)–ρ-normal set. This is indeed the case as is given by the following theorem which is the main result of this paper:

THEOREM 1. *If T is a (p, q)–ρ-normal set, then*

$$\|H_{npq}(f, \mathbf{d}_n, \mathbf{e}, \mathbf{m}) - f\|_I = o(1) \qquad \textit{as } n \to \infty \textit{ for all } f \in C \tag{19}$$

whenever

$$|d_{kn}| = O(n^{\rho+\delta}) \qquad \textit{for arbitrarily small } \delta > 0,\ k=1, ..., n. \tag{20}$$

If T is only (p, q)-normal, then (19) *holds with I replaced by I_ε.*

Proof. This proof follows that of Vértesi [9] which is based on the work of Grünwald [5]. We first observe that we can imporve Theorem 3.1 in [11] to read that if

$$\sum_{k=1}^{n} |h_{knpq}(x)| = O(1) \tag{21}$$

$$\sum_{k=1}^{n} (1+|d_{kn}|)\, |\hat{h}_{knpq}(x)| = o(1) \qquad \text{as} \quad n \to \infty \tag{22}$$

uniformly in I, then (19) holds. The proof of this depends on the fact that, given any $\eta>0$, we can find a polynomial $P_m \in \mathscr{P}_m$ for sufficiently large m such that $\|f-P_m\|_I<\eta$ and $P_m^{(s)}(1)=e_{s+1}$, $s=0, ..., p-1$, $P_m^{(t)}(-1)=m_{t+1}$, $t=0, ..., q-1$. For example, we can take P_m to be $H_{npq}^{(\alpha,\beta)}(f, \mathbf{d}_n, \mathbf{e}, \mathbf{m})$ based on the zeros of $P_N^{(\alpha,\beta)}$ with $\alpha=p-1/2$, $\beta=q-1/2$, and $d_{kN}=0$, $k=1, ..., N$ for sufficiently large N. Now, for $n \geqslant 2N+p+q-1$, $H_{npq}(P_m, P'_m, \mathbf{e}, \mathbf{m})=P_m$. Hence

$$\begin{aligned}
&|H_{npq}(f, \mathbf{d}_n, \mathbf{e}, \mathbf{m}; x)-f(x)| \\
&\quad \leqslant \sum_{k=1}^{n} |f_{kn}-P_m(x_{kn})|\,|h_{knpq}(x)| \\
&\qquad + \sum_{k=1}^{n} (|d_{kn}|+|P'_m(x_{kn})|)\,|\hat{h}_{knpq}(x)| + |P_m(x)-f(x)| = o(1)
\end{aligned}$$

as $n \to \infty$ uniformly in I if (21) and (22) hold. But (21) follows from (14). Hence, to prove our theorem, we must show that (22) holds uniformly in I whenever (20) holds.

Let $\rho_1=\rho-\delta/2$ and $\rho_2=\rho-\delta$ and define the function $g \in C(I)$ as

$$g(x)=\begin{cases} 0, & -1 \leqslant x \leqslant \alpha \\ (x-\alpha)^{\rho_1}, & \alpha \leqslant x \leqslant 1, \end{cases}$$

where $\alpha \in [-1, 0]$. Using g, we shall verify that

$$\sum_{\alpha \leqslant x_{kn}} (x_{kn}-\alpha)^{\rho_1} A_{knpq}(\alpha) \leqslant cn^{-\rho_2}, \tag{23}$$

where we shall use c to indicate an arbitrary positive constant independent of n and x but changing values at each new occurrence. Now, if $\alpha=x_{jn}$ for an index j or if $\alpha=-1$, then the sum in (23) vanishes. Hence, we consider only those values of n for which $x_{kn} \neq \alpha$, $k=1, ..., n$, and assume that $\alpha>-1$. For such values of n and α, we can form

$$\hat{H}_n(x) := H_{npq}(g, g', \hat{\mathbf{e}}, \hat{\mathbf{m}}; x),$$

where $\hat{e}_2=g'(1)$, $\hat{e}_s=0$, $s=3, ..., p$, and $\hat{m}_t=0$, $t=2, ..., q$. We first show that

$$\begin{aligned}
\hat{H}_n(\alpha) = &\sum_{\alpha<x_{kn}} (x_{kn}-\alpha)^{\rho_1}[v_{knpq}(\alpha)-\rho_1]\,A_{knpq}(\alpha) \\
&+ \chi_{0npq}(\alpha)\,g(1)+\chi_{1npq}(\alpha)\,g'(1) \geqslant 0.
\end{aligned} \tag{24}$$

By (p, q)–ρ-normality, the sum in (24) is nonnegative. Furthermore, setting $B(\alpha) := (\omega_n(\alpha)/\omega_n(1))^2((1+\alpha)/2)^q$, we have that

$$\begin{aligned}
&\chi_{0npq}(\alpha)\, g(1) + \chi_{1npq}(\alpha)\, g'(1)\\
&\quad = [V_{0npq}(\alpha)(1-\alpha)^{\rho_1} + V_{1npq}(\alpha)\,\rho_1(1-\alpha)^{\rho_1-1}(1-\alpha)]\, B(\alpha)\\
&\quad = \left[\sum_{\sigma=0}^{p-1} a_{\sigma nq}(1-\alpha)^\sigma - \sum_{\sigma=0}^{p-2} \rho_1 a_{\sigma nq}(1-\alpha)^\sigma\right](1-\alpha)^{\rho_1} B(\alpha)\\
&\quad = \left[\sum_{\sigma=0}^{p-2} (1-\rho_1)\, a_{\sigma nq}(1-\alpha)^\sigma + a_{p-1,nq}(1-\alpha)^{p-1}\right](1-\alpha)^{\rho_1} B_{nq}(\alpha) \geqslant 0
\end{aligned}$$

since $\rho_1 < 1$ and $a_{\sigma nq} \geqslant 0$ by Lemma 1, proving (24).

We now invoke a lemma similar to that in [5] which we shall prove later.

Lemma 2. *For the function $g(x)$, there exist polynomials $P_m \in \mathscr{P}_m$ for all m such that, for all $x \in I$*

$$|g(x) - P_m(x)| \leqslant c \log m/m^{\rho_1} \leqslant cm^{-\rho_2} \tag{25}$$

$$|(x-\alpha)(g'(x) - P'_m(x))| \leqslant c \log m/m^{\rho_1} \leqslant cm^{-\rho_2}, \qquad x \neq \alpha \tag{26}$$

$$|P_m^{(i)}(1)| \leqslant cm^{-\rho_2}/(1-\alpha)^i, \qquad i = 1, ..., p-1,\ \alpha \neq 1 \tag{27}$$

$$|P_m^{(j)}(-1)| \leqslant cm^{-\rho_2}/(1+\alpha)^j, \qquad j = 2, ..., q-1,\ \alpha \neq -1. \tag{28}$$

For such polynomials, setting $M = 2n + p + q - 1$, we have that

$$\begin{aligned}
&|\hat{H}_n(\alpha) - P_M(\alpha)|\\
&\quad \leqslant \sum_{k=1}^{n} |g(x_{kn}) - P_M(x_{kn})|\, h_{knpq}(\alpha)\\
&\qquad + \sum_{k=1}^{n} |g'(x_{kn}) - P'_M(x_{kn})|\, |x_{kn} - \alpha|\, A_{knpq}(\alpha) + \chi_{0npq}(\alpha)\, |g(1) - P_M(1)|\\
&\qquad + |\chi_{1npq}(\alpha)|\, |g'(1) - P'_M(1)| + \sum_{s=2}^{p-1} |\chi_{snpq}(\alpha)\, P_M^{(s)}(1)|\\
&\qquad + \bar{\chi}_{0npq}(\alpha)\, |P_M(-1)| + |\bar{\chi}_{1npq}(\alpha)|\, |P'_M(-1)| + \sum_{t=2}^{q-1} |\bar{\chi}_{tnpq}(\alpha)\, P_M^{(t)}(-1)|\\
&\quad = O(M^{-\rho_2}) + O(M^{-\rho_2}) + O(M^{-\rho_2}) + O(M^{-\rho_2}) + O(M^{-\rho_2})\\
&\qquad + O(M^{-\rho_2}) + O(M^{-\rho_2}) + O(M^{-\rho_2}) = O(M^{-\rho_2}).
\end{aligned}$$

The first, third, and sixth estimates follow from (25) and (14) or (15), the second, from (26) and (16), the fourth from (26) and (17), the fifth from (27) and (17), the seventh from (26) and (18) since $g'(-1)=0$, and the last from (28) and (18).

Since $P_M(\alpha)=O(M^{-\rho_2})$ from (25), we have finally that

$$0 \leqslant \hat{H}_n(\alpha) \leqslant cM^{-\rho_2}. \tag{29}$$

Since $v_{knpq}(\alpha)-\rho_1 \geqslant \rho-\rho_1=\delta/2$, we obtain (23) from (29). As in [9], we have that $(x_{kn}-\alpha)^{\rho_1} \geqslant 2^{\rho_1}(x_{kn}-\alpha)/2$. Then, using similar arguments for $\alpha \in [0, 1]$, we obtain (22) proving (19). The proof of the (p, q)-normal case is similar.

Proof of Lemma 2. Since $g(x)$ satisfies a Hölder condition of order ρ_1, we can find a polynomial $Q_m \in \mathscr{P}_m$ for every m such that

$$|g(x)-Q_m(x)| \leqslant cm^{-\rho_1} \quad \text{in} \quad I \tag{30}$$

and

$$Q_m^{(i)}(1)=0, i=1, ..., p-1; \qquad Q_m^{(j)}(-1)=0, j=1, ..., q-1. \tag{31}$$

For example, we can take $H_{npq}(g, \mathbf{d}_n, \mathbf{e}, \mathbf{m}; x)$ based on the zeros of $P_n^{(\alpha,\beta)}$ with $\alpha=p-1/2$, $\beta=q-1/2$ where $d_{kn}=0$, $k=1, ..., n, e_{s+1}=0$, $s=1, ..., p-1$, and $m_{t+1}=0$, $t=1, ..., q-1$. See in [11, 3.4.3] combined with Lemma 4.2. If we now define

$$P_m(x)=\int_{-1}^{x} \frac{Q_m(t)-Q_m(\alpha)}{t-\alpha}\,dt \tag{32}$$

then, as is shown in [5], (25) and (26) hold. Since

$$P'_m(x)=\frac{Q_m(x)-Q_m(\alpha)}{x-\alpha} \tag{33}$$

and

$$P_m^{(i)}(x)=\frac{(1-i)\,P_m^{(i-1)}(x)+Q^{(i-1)}(x)}{x-\alpha}, \qquad i \geqslant 2 \tag{34}$$

as can be shown by induction, it follows from the fact that $|Q_m(\alpha)|=O(m^{-\rho_1})$ that

$$|P'_m(1)|=|Q_m(1)/(1-\alpha)|+O(m^{-\rho_1}) \leqslant cm^{-\rho_1}/(1-\alpha)$$

$$|P'_m(-1)| \leqslant (|Q_m(-1)|+|Q_m(\alpha)|)/(1+\alpha) \leqslant cm^{-\rho_1}/(1+\alpha).$$

Hence, by (31) and induction, (27) and (28) follow from (34).

3. Concluding Remarks

The (p, q)–ρ-normal sets have many of the properties of ρ-normal sets. We mention here one which may be useful in product integration as a theorem (cf. [5]).

THEOREM 2. *If T is a (p, q)–ρ-normal set, then for any ε in $(0, 1)$ and any $f \in C(I)$,*

$$\left| f(x) - \sum_{k=1}^{n} f_{kn} A_{knpq}(x) \right| \to 0 \qquad \text{as} \quad n \to \infty$$

uniformly for all $x \in I_\varepsilon$.

Proof. The proof proceeds exactly as in [5] with $\omega_n''(x_{kn})/\omega_n'(x_{kn})$ replaced by $A_{knpq}(x)$.

References

1. L. Fejér, Über Interpolation, *Göttinger Nachrichten (1)* (1916), 66–91.
2. L. Fejér, Lagrangesche Interpolation und die zugehörigen konjugierten Punkte, *Math. Ann.* **106** (1932), 1–55.
3. L. Fejér, On the characterization of some remarkable systems of points of interpolation by means of conjugate points, *Amer. Math. Monthly* **41** (1943), 1–14.
4. H. H. Gonska and H.-B. Knoop, On Hermite–Fejér interpolation: A bibliography (1914–1987), *Studia Sci. Math. Hungar.*, in press.
5. G. Grünwald, On the theory of interpolation, *Acta Math.* **75** (1942), 219–245.
6. H.-B. Knoop, "Hermite–Fejér-Interpolation mit Randbedingungen," Habilitationsschrift, Universität Duisburg, 1981.
7. J. Sánta, Convergence theorems of quasi-Hermite Fejér interpolation, *Publ. Math. Debrecen* **22** (1975), 23–29.
8. P. Szász, On quasi-Hermite-Fejér interpolation, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 413–439.
9. P. Vértesi, Hermite–Fejér type interpolations, III, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 67–84.
10. P. Vértesi, ρ-Normal point systems, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 267–277.
11. P. Vértesi, Hermite–Fejér type interpolations, IV (Convergence criteria for Jacobi abscissas), *Acta Math. Acad. Sci. Hungar.* **39** (1982), 83–93.